This week, we’ve been discussing ways to classify problems according to their difficulty, using the notions of polynomial-time reductions and polynomial-time verifiers, and NP-Completeness. In this lab, you’ll develop more sophisticated polynomial-time reductions using gadgets.

Below is a synopsis of relevant decision problems for this lab.

- **SAT.** The input for SAT is a set of $n$ boolean variables $x_1, \ldots, x_n$ and $m$ clauses $c_1, \ldots, c_m$, where each clause is the OR of one or more literals e.g. $c_i = x_1 \lor \overline{x_2} \lor x_3 \lor \overline{x_17}$. Output YES iff there is a truth assignment to $x_1, \ldots, x_n$ that satisfies every clause.

- **3-SAT.** The input for 3-SAT is the same as for SAT, except that each clause is the OR of exactly three literals.

- **Three-Coloring.** The input for Three-Coloring is a graph $G = (V, E)$. Output YES iff the vertices can be colored using three colors such that each edge has different-colored endpoints.

1. In the first exercise, you will reduce 3-Sat $\leq_p$ Three-Coloring. Before getting there, it will be helpful to create some interesting three-colorable graphs. In all of the following exercises, you are to create a three-colorable graph (say the colors are red, blue, green) with certain special properties. The graphs you create should include three vertices marked $a, b, c$ but can (and often will) include other vertices. Except for the properties specified, these vertices should be unconstrained. For example, unless the problem states that e.g. $a$ cannot be red, it must be possible to color the graph in such a way that $a$ is red. (You may fix colors for other vertices, just not $a, b, c$, and not in a way that constrains the colors of $a, b, c$.)

(a) Create a graph such that $a, b, c$ all have different colors.

(b) Create a graph such that $a, b, c$ all have the same color.

(c) Create a graph such that $a, b, c$ do NOT all have the same color.

(d) Create a graph such that none of $a, b, c$ can be green.

(e) Create a graph such that none of $a, b, c$ are green, and they cannot all be blue.

2. Show that 3-Sat $\leq_p$ Three-Coloring. (Hint: Associate the color red with True and the color blue with False.)

3. Show that Three-Coloring $\in$ NP-Complete.

4. Show that Sat $\leq_p$ 3-Sat.

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1A literal is either a boolean variable $x_i$ or its negation $\overline{x_i}$. 

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