The Probabilistic Method
- technique for proving existence of well-behaved good objects
- Good object: object with nice properties.
- Idea:
  - choose object at random,
  - show that Pr[object good]>0
  - Conclude there must exist a good object
- Applications: combinatorics, circuit complexity, communication complexity, …
- Sometimes requires sophisticated arguments, but often probability is basic

**Example: Washer Problem**
- There are 650 points in circle of radius 16 in
- You have washer
  - inner radius 2 in
  - outer radius 3 in

Show you can place washer so it covers 10 points

**Intuition**
Q: consider any point \( p \), where can I place center of washer to cover \( p \)?
A: Anywhere between \( 2-3 \) in away there is region of area \( \pi r_2^2 - \pi r_1^2 = 5\pi \) where I can place washer to cover \( p \).

**Note:** point can be on edge of circle we don’t know where

**Solution**
Place center of washer uniformly in radius 19 in circle.

\[ X_p: \text{ indicator var for event that } p \text{ covered} \]

\[ X_i = \sum_{p} X_p \text{ # points covered} \]
radius 19 circle:  \( \pi \cdot 19^2 = 361 \pi \)

\[
E[x^2] = \mathbb{P}[x \text{ covered}] = \frac{5 \pi}{361 \pi} = \frac{5}{361}
\]

\[
E[x] = E[E[x^2]] = 650.5 \approx 9.0027777 > 9
\]

\[\Rightarrow \mathbb{P}[x > 9] > 0 \quad (\text{otherwise } E[x] \leq 9)\]

\(\Rightarrow\) must be some place for washer that covers \(\geq 10\) pts.

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Example 2

**Graph Version**
- an \( n \)-vertex complete graph
- edges are red or blue
- no monochromatic triangles \( \triangle \) \( \triangle \)

**Def:** \( R(3) \): smallest \( n \) such that every \( n \)-vertex complete graph has a \( \triangle \) or \( \triangle \)

\( R(k) \): smallest \( n \) such that every \( n \)-vertex complete graph contains \( k \)-vertex monochromatic subgraph \( \square \) \( \square \)

**Fact:** \( R(3) = 6 \)

\( \text{Today: } R(k) \geq 2^{k/2} \) \[\text{[Erdős 1947]}\]

**Also Known:**
\[ R(k) \leq \frac{4^k}{k^k} \] \[\text{[Erdős-Szekeres 1935]}\]
\[ R(K) \leq \frac{4^k}{\text{polylog}(k)} \]
\[ R(K) \geq 2^k \]
\[ R(K) \leq (4-\varepsilon)^k \]

Then:
\[ R(K) \geq 2^{\frac{k^2}{2}} \]

- For any set \( S \) of \( k \) vertices
  \[ \text{BAD}_S : \text{event that } S \text{ monochromatic} \]

- \( \text{BAD} = \bigcup S \text{ BAD}_S \)

Note: There are \( \binom{n}{k} \) sets \( S \)

There are \( \binom{n}{k} \cdot 2^\frac{k^2}{2} \) edges \( u, v \) \( u \in S \)

\[ \Pr[\text{BAD}_S] = 2 \cdot \left(\frac{1}{2}\right)^{k^2/2} \]

\[ \Pr[\overline{\text{BAD}}] = \binom{n}{k} \cdot 2 \cdot 2^{-\frac{k^2}{2}} \]
\[ < n^k \cdot 2^{k^2} \]
\[ = 2^{k \log(n) - k^2} \]

2 \( \frac{\varepsilon n k}{k^2} < n^k \)
Note: if \( k \log(n) - \frac{k^2}{2} < 0 \) then \( P[\text{BAD}] < 1 \)

so \( \exists \) \( n \)-vertex graph

with monochromatic \( k \)-vertex subgraph

\[ k \log(n) - \frac{k^2}{2} < 0 \iff \]

\[ k \log(f) < \frac{k^2}{2} \iff \]

\[ \log(n) < \frac{k}{2} \iff \]

\[ n < 2^{\frac{k}{2}} \]

Conclusion: if \( n < 2^{\frac{k}{2}} \) then

possible to color edges red/blue

so no subgraph on 2+ trees

all blue or all red

\[ \Rightarrow R(k) \geq 2^{\frac{k}{2}} \]