Probabilistic Method
- recap technique
- Washer problem
- Ramsey Theory

The Probabilistic Method
- technique for proving existence of well-behaved good objects
- Good object: object with nice properties.
- Idea:
  - choose object at random,
  - show that Pr[object good]>0
  - Conclude there must exist a good object
- Applications: combinatorics, circuit complexity, communication complexity, …
- Sometimes requires sophisticated arguments, but often probability is basic

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Example: Washer Problem

- There are 650 points in circle of radius 16 in
- You have washer  inner radius 2 in, outer radius 3 in

Show you can place washer so it covers 10 points

Intuition

Q: consider any point p. where can I place center of washer to cover p?

A: Any where between 2-3 in away

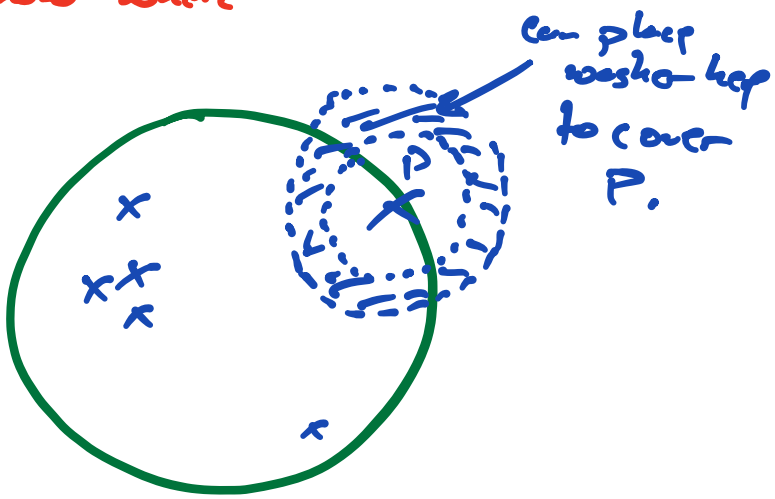
there is region of area $\pi 3^2 - \pi 2^2 = 5\pi$

where I can place washer to cover p.

Note: point can be on edge of circle we don't know where

Solution

Place center of washer uniformly in radius 19 in circle.



X_p: indicator var for event that p covered

$X := \sum_p X_p$ #points covered.

area of radius 19 circle: $\pi \cdot 19^2 = 361\pi$

$$E[X_p] = \Pr[p \text{ covered}] = \frac{5\pi}{361\pi} = \frac{5}{361}$$

$$E[X] = \sum_p E[X_p] = \frac{650 \cdot 5}{361} \approx 9.00277 > 9$$

$\Rightarrow \Pr[X > 9] > 0$ (otherwise $E[X] \le 9$)

$\Rightarrow$ must be some place for washer that covers ≥ 10 pts

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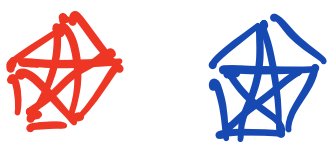
Example 2

Graph Version – n vertex complete graph
– edges are red or blue
– no monochromatic triangles △ △

Defn $R(3)$: smallest n such that every n-vertex complete graph has a △ or △

$R(k)$: smallest n such that every n-vertex complete graph contains k-vertex monochromatic subgraph ⊠ ⊠

Fact: $R(3) = 6$



Today: $R(k) \ge 2^{k/2}$ [Erdős 1947]

Also known:

$$R(k) \le \frac{4^k}{\sqrt{k}}$$

[Erdős-Szekeres 1935]

$R(k) \leq \frac{4^k}{2^{\text{poly}(\log(k))}}$ [Conlon 2009]

$R(k) \geq k \cdot 2^{k/2}$ [Spencer 1975]

$R(k) \leq (4-\varepsilon)^k$ [Campos et al March 2023]

Thm $R(k) \geq 2^{k/2}$.

- For any set S of k vertices
  BAD_S : event that S monochromatic
- $BAD = \bigcup_S BAD_S$

Note: There are $\binom{n}{k}$ sets S

There are $\binom{k}{2} \approx \frac{k^2}{2}$ edges (u,v) w/ $u,v \in S$

$$\Pr[BAD_S] = 2 \cdot \left(\frac{1}{2}\right)^{k^2/2}$$

$$\Pr[\overline{BAD}] = \binom{n}{k} \cdot 2 \cdot 2^{-\frac{k^2}{2}}$$

$$< n^k \cdot 2^{-k^2/2}$$

$$= 2^{k\log(n) - k^2/2}$$

$\binom{n}{k} \leq \frac{2n^k}{k!} < n^k$

Note: if $k\log(n) - \frac{k^2}{2} < 0$ then $\Pr[\text{BAD}] < 1$

so $\exists$ n-vertex graph w/o monochromatic k-vertex subgraph

$k\log(n) - \frac{k^2}{2} < 0 \iff$

$k\log(n) < \frac{k^2}{2} \iff$

$\log(n) < \frac{k}{2} \iff$

$n < 2^{k/2}$

Conclusion: if $n < 2^{k/2}$ then

possible to color edges red/blue

so no subgraph on k vertices all blue or all red

$\Rightarrow$ $\boxed{R(k) \geq 2^{k/2}}$