CS91T Lab Assignment 5

This homework is a one-week assignment, due at **11:59PM on Wednesday, April 12.** This is a one-week, ten point assignment. Write your solution using LaTeX. Submit this homework in a file named `hw5.tex` using **github**.

This is a partnered assignment, but you may choose to work on this solo, assuming no students in the class need a partner. You should primarily be discussing this assignment with your partner. It's ok to discuss approaches at a high level. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

The main **learning goals** of this assignment are to gain practice working with the Second Moment Method.

1. **(Shoup, exercise 8.26b.)** The *covariance* of real-valued random variables X, Y, written $\textsc{Cov}[X, Y]$, is defined as:

   $$\textsc{Cov}[X, Y] = E[XY] - E[X]E[Y] \ .$$

   For real-valued random variables $X_1, \ldots, X_n$, show that $\textsc{Var}[\sum_i X_i] = \sum_i \textsc{Var}[X_i] + \sum_{i \neq j} \textsc{Cov}[X, Y]$.

2. **Threshold theorem for Connectivity.** In class and lab, we sketched out details for proving

   Theorem 1. *Let $p = \frac{c \ln(n)}{n}$ for some constant $c > 0$, and let $G \sim G_{n,p}$. If $c < 1$, then with high probability G is not connected.*

   In particular, we defined $A = \sum_i A_i$, where each A_i is the indicator variable for the event that vertex i is *isolated*. We saw that $E[A_i] \approx n^{-c}$, $E[A] \approx n^{1-c}$, and using the Second Moment Method,

   $$\Pr[A = 0] \leq \frac{\textsc{Var}[A]}{E[A]^2} \ .$$

   (a) Compute $\textsc{Var}[A_i]$.

   (b) Using Exercise 1, show that $\textsc{Var}[A] \leq 2n^{1-c}$.

   (c) Using the Second Moment Method, conclude that, for large enough n,

   $$\Pr[A = 0] \leq \frac{8}{n^{1-c}} = O\left(\frac{1}{n^{1-c}}\right) \ .$$

3. **(extra challenge problem).** Prove upper bound on the connectivity theshold theoremm:

   Theorem 2. *Let $p = \frac{c \ln(n)}{n}$ for some constant $c > 0$, and let $G \sim G_{n,p}$. If $c > 1$, then* $\Pr[G \textit{ is NOT connected}] \leq \frac{1}{\text{poly}(n)}$.