CS91T Lab Assignment 5

This homework is a one-week assignment, due at 11:59PM on Wednesday, April 12. This is a one-week, ten point assignment. Write your solution using \LaTeX. Submit this homework in a file named \texttt{hw5.tex} using \texttt{github}.

This is a partnered assignment, but you may choose to work on this solo, assuming no students in the class need a partner. You should primarily be discussing this assignment with your partner. It’s ok to discuss approaches at a high level. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

The main learning goals of this assignment are to gain practice working with the Second Moment Method.

1. \textbf{(Shoup, exercise 8.26b.)} The covariance of real-valued random variables \(X, Y\), written \(\text{Cov}[X, Y]\), is defined as:

\[
\]

For real-valued random variables \(X_1, \ldots, X_n\), show that
\[
\text{Var}\left[\sum_i X_i\right] = \sum_i \text{Var}[X_i] + \sum_{i \neq j} \text{Cov}[X, Y].
\]

2. \textbf{Threshold theorem for Connectivity.} In class and lab, we sketched out details for proving

\textbf{Theorem 1.} Let \(p = \frac{c \ln(n)}{n}\) for some constant \(c > 0\), and let \(G \sim G_{n,p}\). If \(c < 1\), then with high probability \(G\) is not connected.

In particular, we defined \(A = \sum_i A_i\), where each \(A_i\) is the indicator variable for the event that vertex \(i\) is isolated. We saw that \(E[A_i] \approx n^{-c}\), \(E[A] \approx n^{1-c}\), and using the Second Moment Method,

\[
\text{Pr}[A = 0] \leq \frac{\text{Var}[A]}{E[A]^2}.
\]

(a) Compute \(\text{Var}[A_i]\).

(b) Using Exercise 1, show that \(\text{Var}[A] \leq 2n^{1-c}\).

(c) Using the Second Moment Method, conclude that, for large enough \(n\),

\[
\text{Pr}[A = 0] \leq \frac{8}{n^{1-c}} = O\left(\frac{1}{n^{1-c}}\right).
\]

3. \textbf{(extra challenge problem).} Prove upper bound on the connectivity threshold theorem:

\textbf{Theorem 2.} Let \(p = \frac{c \ln(n)}{n}\) for some constant \(c > 0\), and let \(G \sim G_{n,p}\). If \(c > 1\), then
\[
\text{Pr}[G \text{ is NOT connected}] \leq \frac{1}{\text{poly}(n)}.
\]