CS91T Lab Assignment 2

This homework is due at 11:59PM on Wednesday, February 22. Write your solution using LaTeX. Submit this homework using **github**.

This is a partnered assignment. You should primarily be discussing this assignment with your partner. It's ok to discuss approaches at a high level. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

When you submit homework assignments this semester, please keep the following in mind:

- Don't forget to fill out the **homework submission poll** using Pollster.
- Don't submit a .pdf – just the .tex will do.
- I will compile your .tex file using pdflatex. It is your responsibility to make sure the LaTeX compiles.

The main **learning goals** of this assignment are to gain more practice with discrete probabilty and to think more deeply about randomized algorithms.

1. We've modeled the randomness in a randomized algorithm as a single operation flipCoin(), which takes no input and returns 1 with probability 0.5 and 0 with probability 0.5. However, in reality it might be very difficult to find a source of randomness that produces truly uniform random bits.

   In particular, suppose that instead of flipCoin(), you're given an operation bogoCoin(), which also takes no inputs, but returns 1 with probability p and 0 with probability $1-p$, for some $0 < p < 1$. Furthermore, suppose that the precise value of p is *unknown* to you. How can you use bogoCoin() to generate uniform random bits?

   (a) Design an implementation of flipCoin() that uses bogoCoin() as a subroutine.

   (b) Argue that your implementation of flipCoin() works; i.e., that the bit it returns is uniform.

   (c) What is the expected number of calls to bogoCoin() that your algorithm makes? Express your answer in terms of p.

2. A random variable $X \in \{0, 1\}$ has bias δ for some $0 \leq \delta \leq 1$ if either $\Pr[X = 1] = \frac{1}{2}(1 + \delta)$ or $\Pr[X = 1] = \frac{1}{2}(1 - \delta)$. In the former case, we say that X is *biased towards 1.* In the latter case, X is *biased towards 0.* For example, if $X == 1$ with probabiltiy 3/4, then X has bias 1/2 since $\frac{3}{4} = \frac{1}{2}(1 + \frac{1}{2})$.

   Now, suppose you have n random bits $X_1, \ldots, X_n$, and let Y be the XOR of the random bits:

   $$Y := X_1 \oplus X_2 \oplus \cdots \oplus X_n \ .$$

   Show that if the bias of each X_i is δ_i, then the bias of Y is $\delta_1 \delta_2 \cdots \delta_n$.

   Hint: Prove this by induction on n.

3. Recall the EQUALITY problem from class: Alice and Bob are given n bit strings $x, y \in \{0,1\}^n$ and must communicate to decide whether or not x equals y.

   In class we saw a strategy that used seven bits of communication and had error 1/128. Suppose even this error isn't good enough. Give an algorithm for EQUALITY that has error $1/k$. Express the communication cost as a function of k.

4. **Spot the Difference.** In this problem, Alice and Bob again have n-bit strings $x, y \in \{0,1\}^n$. This time, they know the strings are equal, except in one bit; i.e., there is a unique index $1 \leq i \leq n$ such that $x_i \neq y_i$. Define $\text{SPOTTD}(x, y) := i$ to be this unique location.

   Give a randomized communication protocol for Alice and Bob to compute $\text{SPOTTD}(x, y)$. What is the communication cost of your protocol? You will earn full credit for any protocol that uses $O(log(n))$ bits.

5. **Probabilistic Method.** Suppose there are 650 points in a circle of radius 16 inches. You have a washer of outer radius 3 inches, inner radius 2 inches.

   Show you can place the washer so it covers at least 10 points.