CS46 Lab 9

Lab problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions.** The purpose of these lab problems is to get more comfortable with reasoning and writing about Turing machines, decidability, and recognizability.

Note: There is more than 90 minutes of exercises on this lab. Do not feel obligated to solve these exercises in a linear fashion. Work on the problems that are most interesting to your group.

1. Prove the following theorem.

   Theorem. *A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable.*

2. Let $L_{TM} = \{\langle M \rangle | M \text{ is an encoding of a Turing machine}\}$.

   - Show that L_{TM} is decidable.
   - Replace "Turing machine" with any of DFA, NFA, CFG, PDA. Show that the problem remains decidable.

3. **Useless variables.** Given a grammar G, we say that a variable $V \in G$ is "useless" if there is no string w for which a possible derivation of w contains the variable V. Formulate the problem of finding grammars containing useless variables as a language and show that this language is decidable.

4. **Infinite languages.** Last week we saw that the following language was decidable:

   $$\mathit{INFINITE}_{\mathrm{DFA}} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) \text{ is an infinite language}\}$$

   (See solved problem 4.10 in the book for a clever way of making this argument.)

   (a) Show that $\mathit{INFINITE}_{\mathrm{CFG}}$ is decidable, where:

   $$\mathit{INFINITE}_{\mathrm{CFG}} = \{\langle G \rangle \mid G \text{ is a context-free grammar and } L(G) \text{ is an infinite language}\}$$

   (b) Show that $\mathit{INFINITE}_{\mathrm{TM}}$ is not decidable, where:

   $$\mathit{INFINITE}_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is an infinite language}\}$$

5. **Classifying languages.** For each of the following languages, is the language decidable? Turing-recognizable? co-Turing-recognizable?

   Provided an argument for your answers. (Give the deciders/recognizers that you claim exist, and show why they work; if they do not exist, then prove why not.)

   (a) $E_{\mathrm{TM}} = \{\langle M \rangle \mid L(M) = \emptyset\}$

   (b) $\mathit{HUNDRED}_{\mathrm{TM}} = \{\langle M, w \rangle \mid M \text{ never moves its head past the } 100^{\text{th}} \text{ tape square during its computation on } w\}$

6. In class we saw a proof that A_{TM} is undecidable. How does this proof use the fact that we're working with Turing Machines?

   - Suppose we wanted to mimic the proof to show that A_{DFA} is undecidable (it is not). Begin by replacing each occurence of a Turing Machine with a DFA, and see what happens. Where does the proof break down?
   - Perhaps the proof breaks down not in the TM vs DFA part, but in the decidability part. Try to mimic the proof that A_{TM} is undecidable to show that A_{DFA} is not regular.

   Note: this problem is less concrete than some of the other problems we've encountered. Do not expect clean, crisp answers. Instead, use this problem as a discussion point as you try to determine how the proof that A_{TM} is undecidable specifically uses that we're working with Turing Machine descriptions instead of DFA deccriptions.

7. **Turing Machines with unary alphabets.** An alphabet is *unary* if it consists of just one character. In this problem, we'll use a unary alphabet $\Sigma = \{a\}$. The purpose of this problem is to explore whether unary encodings of DFAs (or NFAs or ...) are possible and whether they are useful.

   - In class, we saw an encoding of an arbitrary DFA using the alphabet $\{(,),',',0,1,2,3,4,5,6,7,8,9\}$ (call this the "class encoding".) Give an encoding of an arbitrary DFA using Σ.
   - Describe a two-tape Turing Machine T which takes as input a unary encoding of an arbitrary DFA $\langle M \rangle$ and writes out the class encoding of M on the second tape.
   - Let $B_{DFA} = \{\langle M, w \rangle |$ M is a unary encoding of a DFA, $w \in \{a,b\}^*$, and M accepts w$\}$. Show that B_{DFA} is decidable.