CS46 Lab 9

Lab problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions**. The purpose of these lab problems is to get more comfortable with reasoning and writing about Turing machines, decidability, and recognizability.

**Note:** There is more than 90 minutes of exercises on this lab. Do not feel obligated to solve these exercises in a linear fashion. Work on the problems that are most interesting to your group.

1. Prove the following theorem.

   **Theorem.** A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable.

2. Let \( L_{TM} = \{ \langle M \rangle | M \text{ is an encoding of a Turing machine} \} \).
   - Show that \( L_{TM} \) is decidable.
   - Replace "Turing machine" with any of DFA, NFA, CFG, PDA. Show that the problem remains decidable.

3. **Useless variables.** Given a grammar \( G \), we say that a variable \( V \in G \) is “useless” if there is no string \( w \) for which a possible derivation of \( w \) contains the variable \( V \). Formulate the problem of finding grammars containing useless variables as a language and show that this language is decidable.

4. **Infinite languages.** Last week we saw that the following language was decidable:

   \[
   \text{INFINITE}_{\text{DFA}} = \{ \langle A \rangle | A \text{ is a DFA and } L(A) \text{ is an infinite language} \}
   \]

   (See solved problem 4.10 in the book for a clever way of making this argument.)

   (a) Show that \( \text{INFINITE}_{\text{CFG}} \) is decidable, where:

   \[
   \text{INFINITE}_{\text{CFG}} = \{ \langle G \rangle | G \text{ is a context-free grammar and } L(G) \text{ is an infinite language} \}
   \]

   (b) Show that \( \text{INFINITE}_{\text{TM}} \) is not decidable, where:

   \[
   \text{INFINITE}_{\text{TM}} = \{ \langle M \rangle | M \text{ is a Turing machine and } L(M) \text{ is an infinite language} \}
   \]

5. **Classifying languages.** For each of the following languages, is the language decidable? Turing-recognizable? co-Turing-recognizable? Provided an argument for your answers. (Give the deciders/recognizers that you claim exist, and show why they work; if they do not exist, then prove why not.)

   (a) \( E_{TM} = \{ \langle M \rangle | L(M) = \emptyset \} \)

   (b) \( HUNDRED_{TM} = \{ \langle M, w \rangle | M \text{ never moves its head past the 100th tape square during its computation on } w \} \)
6. In class we saw a proof that $A_{TM}$ is undecidable. How does this proof use the fact that we’re working with Turing Machines?

- Suppose we wanted to mimic the proof to show that $A_{DFA}$ is undecidable (it is not). Begin by replacing each occurrence of a Turing Machine with a DFA, and see what happens. Where does the proof break down?
- Perhaps the proof breaks down not in the TM vs DFA part, but in the decidability part. Try to mimic the proof that $A_{TM}$ is undecidable to show that $A_{DFA}$ is not regular.

**Note:** this problem is less concrete than some of the other problems we’ve encountered. Do not expect clean, crisp answers. Instead, use this problem as a discussion point as you try to determine how the proof that $A_{TM}$ is undecidable specifically uses that we’re working with Turing Machine descriptions instead of DFA descriptions.

7. **Turing Machines with unary alphabets.** An alphabet is *unary* if it consists of just one character. In this problem, we’ll use a unary alphabet $\Sigma = \{a\}$. The purpose of this problem is to explore whether unary encodings of DFAs (or NFAs or ...) are possible and whether they are useful.

- In class, we saw an encoding of an arbitrary DFA using the alphabet $\{(, )',', 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (call this the "class encoding"). Give an encoding of an arbitrary DFA using $\Sigma$.
- Describe a two-tape Turing Machine $T$ which takes as input a unary encoding of an arbitrary DFA $\langle M \rangle$ and writes out the class encoding of $M$ on the second tape.
- Let $B_{DFA} = \{(M, w) | M$ is a unary encoding of a DFA, $w \in \{a, b\}^*$, and $M$ accepts $w\}$. Show that $B_{DFA}$ is decidable.