Problem 1: Use the substitution method to show that the recurrence

$$T(n) = \begin{cases} \sqrt{n} \cdot T(\sqrt{n}) + an & \text{if } n < 2 \\ 1 & \text{otherwise} \end{cases}$$

where $a > 0$ has the solution $T(n) = O(n \lg \lg n)$.

Problem 2: Recall that we can quick-sort an array of n elements in $O(n \lg n)$ time if we use an $O(n)$ time partition algorithm to partition an array of n elements into two equally sized arrays such that all elements in one array are smaller than all elements in the other array. We use linear time selection to find the median in $O(n)$ time and partition around this median.

Describe an $O(n \log k)$ time k-partitioning algorithm that partitions an array A of n elements into k equally sized arrays $A_1, A_2, \ldots, A_k$ such that elements in array A_i are smaller than elements in array A_{i+1} for all $i, 1 \leq i < k$. You may assume that k is a power of two (meaning you can ignore floors and ceilings.)

Problem 3: The median of a *sorted* list $L = x_1, x_2, \ldots, x_n$ of n integers is the element with index $\lceil \frac{n}{2} \rceil$ and can be found in $O(1)$ time.

Given two *sorted* lists L_1 and L_2 of n integers each, the median of all $2n$ elements can be found in $O(n)$ time by merging the two lists and returning the element with index n. Design a faster algorithm and bound its asymptotic running time.

Problem 4: Consider two *balanced* search trees T_1 and T_2 representing two sets of integers S_1 and S_2 with n_1 and n_2 elements, respectively. Consider the problem of determining if $S_1 \subseteq S_2$ using T_1 and T_2.

- Describe an $O(n_1 \log n_2)$ time algorithm for determining if $S_1 \subseteq S_2$ using $O(1)$ extra space.
- Describe an $O(n_1 + n_2)$ time algorithm for determining if $S_1 \subseteq S_2$ using $O(n_1 + n_2)$ extra space.
- Describe an $O(n_1 + n_2)$ time algorithm for determining if $S_1 \subseteq S_2$ using $O(\log n_1 + \log n_2)$ extra space.

Problem 5: In this problem we design a divide-and-conquer algorithm for computing the skyline of a set of n buildings.

A *building* B_i is represented as a triplet $(\mathbf{L_i}, H_i, \mathbf{R_i})$ where $\mathbf{L_i}$ and $\mathbf{R_i}$ denote the left and right x coordinates of the building, and H_i denotes the height of the building (note that the x coordinates are drawn boldfaced.)

A *skyline* of a set of n buildings is a list of x coordinates and the heights connecting them arranged in order from left to right (note that the list is of length at most $4n$).

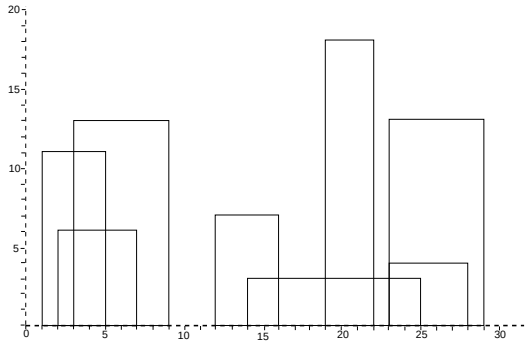
Example: The skyline of the buildings

$$\{(\mathbf{3}, 13, \mathbf{9}), (\mathbf{1}, 11, \mathbf{5}), (\mathbf{12}, 7, \mathbf{16}), (\mathbf{14}, 3, \mathbf{25}), (\mathbf{19}, 18, \mathbf{22}), (\mathbf{2}, 6, \mathbf{7}), (\mathbf{23}, 13, \mathbf{29}), (\mathbf{23}, 4, \mathbf{28})\}$$

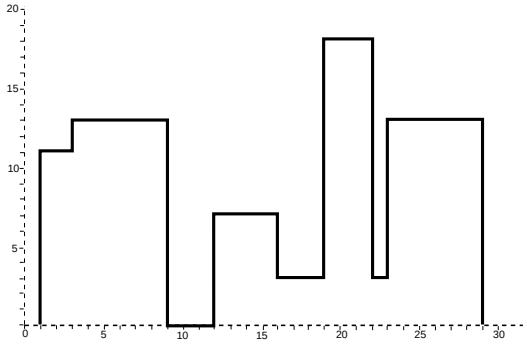
is

$$\{\mathbf{1}, 11, \mathbf{3}, 13, \mathbf{9}, 0, \mathbf{12}, 7, \mathbf{16}, 3, \mathbf{19}, 18, \mathbf{22}, 3, \mathbf{23}, 13, \mathbf{29}, 0\}$$

(note that the x coordinates in a skyline are sorted).



buildings



skyline

1. Let the size of a skyline be the total number of elements (coordinates and heights) in its list.

   Describe an algorithm for combining a skyline A of size n_1 and a skyline B of size n_2 into one skyline S of size $O(n_1+n_2)$. Your algorithm should run in time $O(n_1 + n_2)$.

2. Describe an $O(n \log n)$ algorithm for finding the skyline of n buildings.