# CS 31: Intro to Systems C Programming

## L04: Binary Arithmetic

Vasanta Chaganti & Kevin Webb

Swarthmore College

September 14, 2023

# Announcements

- Clickers will count for credit from this week

# Reading Quiz

- Note the red border!
- 1 minute per question
- No talking, no laptops, phones during the quiz

<u>Check your frequency:</u>

- Iclicker2: frequency AA
- Iclicker+: green light next to selection

For new devices this should be okay,
For used you may need to reset frequency

Reset:
1. hold down power button until
   blue light flashes (2secs)
2. Press the frequency code: AA
   vote status light will indicate success

# Agenda

Data representation & Binary Arithmetic

- number systems + conversion
- sizes, representation
- signedness
- binary arithmetic
- overflow rules

# Abstraction

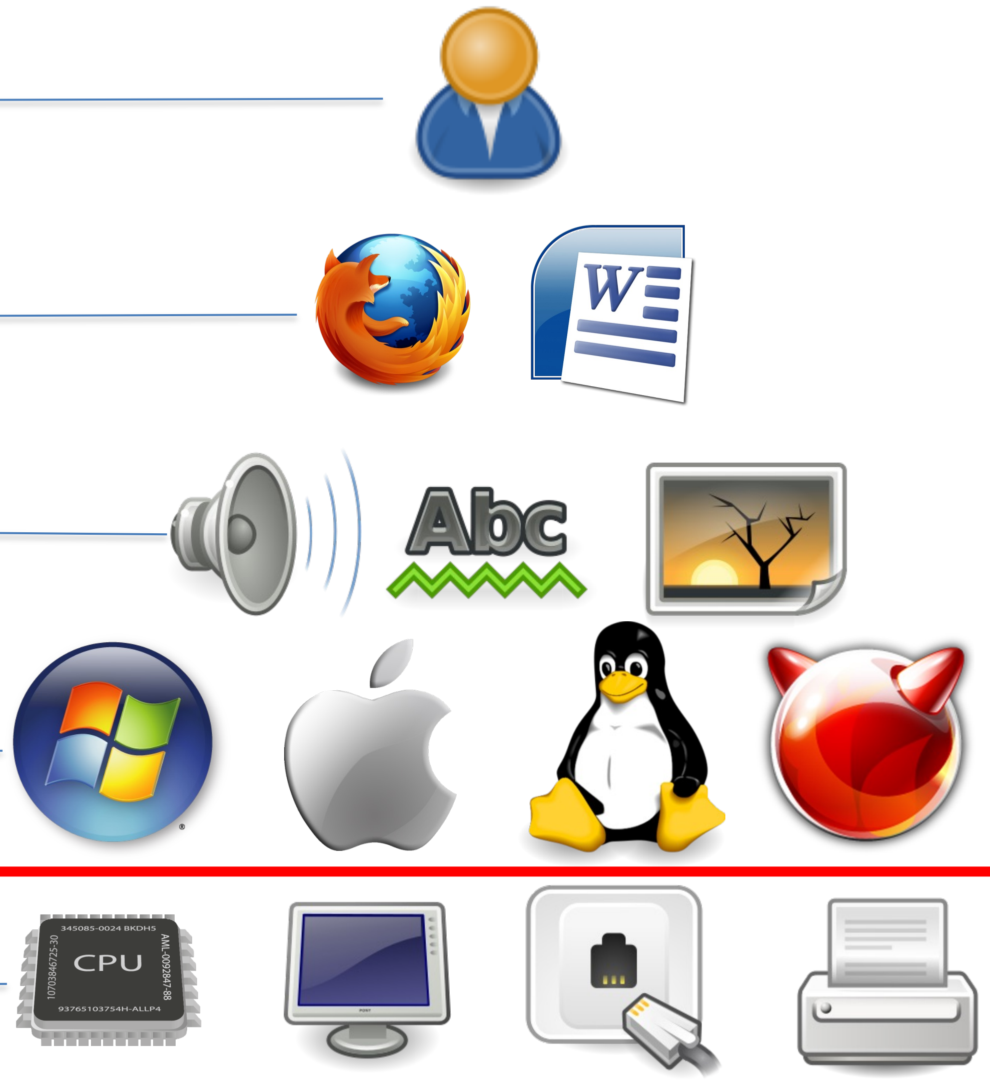
User / Programmer
Wants low complexity

Applications
Specific functionality

Software library
Reusable functionality

Operating system
Manage resources

Complex devices
Compute & I/O



# Bits and Bytes

- Bit: a 0 or 1 value (binary)
  - Hardware represents as two different voltages
    - 1: the presence of voltage (high voltage)
    - 0: the absence of voltage (low voltage)

- Byte: 8 bits, the smallest addressable unit

| | | | | |
|---|---|---|---|---|
| Memory: | 01010101 | 10101010 | 00001111 | … |
| (address) | [0] | [1] | [2] | … |

# Files

Sequence of bytes… nothing more, nothing less

# Binary Digits (BITs)

- One bit: two values (0 or 1)
- Two bits: four values (00, 01, 10, or 11)
- Three bits: eight values (000, 001, …, 110, 111)

# How many values?

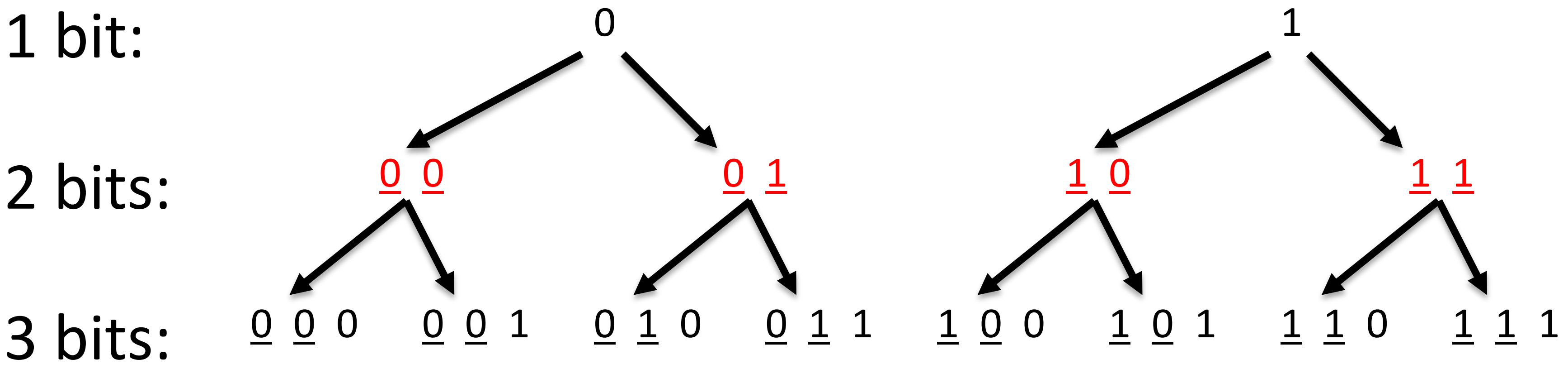
1 bit: 0 1

# How many values?

1 bit: 0 1

2 bits: 0 0 | 0 1 | 1 0 | 1 1

# How many values?



# How many values?

1 bit: 0 1

2 bits: 0 0 0 1 1 0 1 1

3 bits: 0 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1

4 bits:

| | | | | |
|---|---|---|---|---|
| 0 0 0 0 | 0 0 0 1 | 0 0 1 0 | 0 0 1 1 | 16 values |
| 0 1 0 0 | 0 1 0 1 | 0 1 1 0 | 0 1 1 1 | |
| 1 0 0 0 | 1 0 0 1 | 1 0 1 0 | 1 0 1 1 | |
| 1 1 0 0 | 1 1 0 1 | 1 1 1 0 | 1 1 1 1 | |

N bits: $2^N$ values

# Let's start with what we know…

- Digits 0-9
- Positional numbering
- Digits are composed to make larger numbers
- Known as Base 10 representation

# Decimal number system (Base 10)

- Sequence of digits in range [0, 9]

64025

Digit #0: 1’s place, “least significant digit”

Digit #1: 10’s place

Digit #4: “most significant digit”

# Decimal: Base 10

A number, written as the sequence of N digits,

$$d_{n-1} \ldots d_2\, d_1\, d_0$$

where d is in {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, represents the value:

$$[d_{n-1} * 10^{n-1}] + [d_{n-2} * 10^{n-2}] + \ldots + [d_1 * 10^1] + [d_0 * 10^0]$$

64025 =

$6 * 10^4 \quad + \quad 4 * 10^3 \quad + \quad 0 * 10^2 \quad + \quad 2 * 10^1 \quad + \quad 5 * 10^0$

60000 + 4000 + 0 + 20 + 5

# Binary: Base 2

- Used by computers to store digital values.

- Indicated by prefixing number with **0b**

- A number, written as the sequence of N digits, $d_{n-1} \ldots d_2 d_1 d_0$, where d is in {0,1}, represents the value:

$$[d_{n-1} * 2^{n-1}] + [d_{n-2} * 2^{n-2}] + \ldots + [d_2 * 2^2] + [d_1 * 2^1] + [d_0 * 2^0]$$

# Converting Binary to Decimal

Most significant bit → $\underline{1}000111\underline{1}$ ← Least significant bit

7 6 5 4 3 2 1 0

Representation: $1 \times 2^7 + 0 \times 2^6 \ldots + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$128 + \quad\quad + 8 + 4 + 2 + 1$

10001111 = 143

# Hexadecimal: Base 16

Indicated by prefixing number with **0x**

A number, written as the sequence of N digits,

$$d_{n-1} \ldots d_2 d_1 d_0,$$

where d is in {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}, represents:

$$[d_{n-1} * 16^{n-1}] + [d_{n-2} * 16^{n-2}] + \ldots + [d_2 * 16^2] + [d_1 * 16^1] + [d_0 * 16^0]$$

# Generalizing: Base b

The meaning of a digit depends on its position in a number.

A number, written as the sequence of N digits,

$$d_{n-1} \ldots d_2\, d_1\, d_0$$

in base b represents the value:

$$[d_{n-1} * b^{n-1}] + [d_{n-2} * b^{n-2}] + \ldots + [d_2 * b^2] + [d_1 * b^1] + [d_0 * b^0]$$

Base 10: $[d_{n-1} * 10^{n-1}] + [d_{n-2} * 10^{n-2}] + \ldots + [d_1 * 10^1] + [d_0 * 10^0]$

# Other (common) number systems.

- Base 2: How data is stored in hardware.
- Base 8: Used to represent file permissions.
- Base 10: Preferred by people.
- Base 16: Convenient for representing memory addresses.
- Base 64: Commonly used on the Internet, (e.g. email attachments).

It's **all** stored as binary in the computer.

Different representations (or visualizations) of the same information!

# What is the value of 0b110101 in decimal?

A number, written as the sequence of N digits $d_{n-1}...d_2d_1d_0$ where d is in {0,1}, represents the value:

$$[d_{n-1} * 2^{n-1}] + [d_{n-2} * 2^{n-2}] + ... + [d_2 * 2^2] + [d_1 * 2^1] + [d_0 * 2^0]$$

A. 26
B. 53
C. 61
D. 106
E. 128

# What is the value of 0b110101 in decimal?

A number, written as the sequence of N digits $d_{n-1} \ldots d_2 d_1 d_0$ where d is in {0,1}, represents the value:

$$[d_{n-1} * 2^{n-1}] + [d_{n-2} * 2^{n-2}] + \ldots + [d_2 * 2^2] + [d_1 * 2^1] + [d_0 * 2^0]$$

A. 26
B. 53
C. 61
D. 106
E. 128

# What is the value of 0x1B7 in decimal?

$[d_{n-1} * 16^{n-1}] + [d_{n-2} * 16^{n-2}] + \ldots + [d_2 * 16^2] + [d_1 * 16^1] + [d_0 * 16^0]$

(Note: $16^2 = 256$)

A. 397
B. 409
C. 419
D. 437
E. 439

| DEC | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| HEX | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |

# What is the value of 0x1B7 in decimal?

$[d_{n-1} * 16^{n-1}] + [d_{n-2} * 16^{n-2}] + \ldots + [d_2 * 16^2] + [d_1 * 16^1] + [d_0 * 16^0]$

(Note: $16^2 = 256$)

A. 397
B. 409
C. 419
D. 437
E. 439

$1*16^2 + 11*16^1 + 7*16^0 =$

$256 + 176 + 7 = \underline{439}$

| DEC | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| HEX | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |

# Important Point…

- You can represent the same value in a variety of number systems or bases.

- It's **all** stored as binary in the computer.
  - Presence/absence of voltage.

# Hexadecimal: Base 16

- Fewer digits to represent same value
  - Same amount of information!

- Like binary, the base is power of 2

- Each digit is a “nibble”, or half a byte.

# Each hex digit is a “nibble”

- One hex digit: 16 possible values (0-9, A-F)
- $16 = 2^4$, so each hex digit has exactly four bits worth of information.
- We can map each hex digit to a four-bit binary value.
  (helps for converting between bases)

# Each hex digit is a "nibble"

Example value: 0x1B7

Four-bit value: 1

Four-bit value: B (decimal 11)

Four-bit value: 7

In binary: 0001 1011 0111

1 B 7

# Hexadecimal ↔️ Binary Conversion

- Bit patterns as base-16 numbers
- Convert binary to hexadecimal: by splitting into groups of 4 bits each.

Example:

0b0011 1100 1010 1101 1011 0011 = 0x3CADB3

| Bin | 0011 | 1100 | 1010 | 1101 | 1011 | 0011 |
|---|---|---|---|---|---|---|
| Hex | 3 | C | A | D | B | 3 |

# Converting Decimal -> Binary

- Two methods:
  - division by two remainder
  - powers of two and subtraction

Method 1: decimal value D, binary result b ($b_i$ is ith digit):

```
i = 0
while (D > 0)
   if D is odd
          set b_i to 1
   if D is even
          set b_i to 0
   i++
   D = D/2
```

Example: Converting 105

idea:

example: D = 105

$b_0 = 1$

Method 1: decimal value D, binary result b ($b_i$ is ith digit):

```
i = 0
while (D > 0)
   if D is odd
          set b_i to 1
   if D is even
          set b_i to 0
   i++
   D = D/2
```

Example: Converting 105

```
idea:     D              example: D = 105        b_0 = 1
          D = D/2                 D =  52        b_1 = 0
```

Method 1: decimal value D, binary result b ($b_i$ is ith digit):

```
i = 0
while (D > 0)
   if D is odd
          set b_i to 1
   if D is even
          set b_i to 0
   i++
   D = D/2
```

**Example: Converting 105**

| idea: | | example: | | |
|---|---|---|---|---|
| | D | | D = 105 | $b_0$ = 1 |
| | D = D/2 | | D = 52 | $b_1$ = 0 |
| | D = D/2 | | D = 26 | $b_2$ = 0 |
| | D = D/2 | | D = 13 | $b_3$ = 1 |
| | D = D/2 | | D = 6 | $b_4$ = 0 |
| | D = D/2 | | D = 3 | $b_5$ = 1 |
| | D = D/2 | | D = 1 | $b_6$ = 1 |
| | D = 0 (done) | | D = 0 | $b_7$ = 0 |

105 = 01101001

# Method 2

- $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$
- 

  To convert 105:
  - Find largest power of two that's less than 105 (64)
  - Subtract 64 (105 – 64 = 41), put a 1 in $d_6$
  - Subtract 32 (41 – 32 = 9), put a 1 in $d_5$
  - Skip 16, it's larger than 9, put a 0 in $d_4$
  - Subtract 8 (9 – 8 = 1), put a 1 in $d_3$
  - Skip 4 and 2, put a 0 in $d_2$ and $d_1$
  - Subtract 1 (1 – 1 = 0), put a 1 in $d_0$ (Done)

| 1 | 1 | 0 | 1 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|
| $d_6$ | $d_5$ | $d_4$ | $d_3$ | $d_2$ | $d_1$ | $d_0$ |

# What is the value of 357 in binary?

8 7654 3210

digit position

A. 1 0110 0011
B. 1 0110 0101
C. 1 0110 1001
D. 1 0111 0101
E. 1 1010 0101

$2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$,

$2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$

# What is the value of 357 in binary?

8 7654 3210

digit position

A. 1 0110 0011
B. 1 0110 0101
C. 1 0110 1001
D. 1 0111 0101
E. 1 1010 0101

$357 - 256 = 101$
$101 - 64 = 37$
$37 - 32 = 5$
$5 - 4 = 1$

| $d_8$ | $d_7$ | $d_6$ | $d_5$ | $d_4$ | $d_3$ | $d_2$ | $d_1$ | $d_0$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |

$2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$,

$2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$

# So far: Unsigned Integers

<u>With N bits, can represent values: 0 to $2^n$-1</u>

We can always add 0’s to the front of a number without changing it:

10110= $\underline{0}10110$ = $\underline{000}10110$ = $\underline{00000}10110$

# So far: Unsigned Integers

With N bits, can represent values: 0 to $2^n$-1

- `1 byte:  char, unsigned char`
- `2 bytes: short, unsigned short`
- `4 bytes: int, unsigned int, float`
- `8 bytes: long long, unsigned long long, double`
- `4 or 8 bytes: long, unsigned long`

# Unsigned Integers

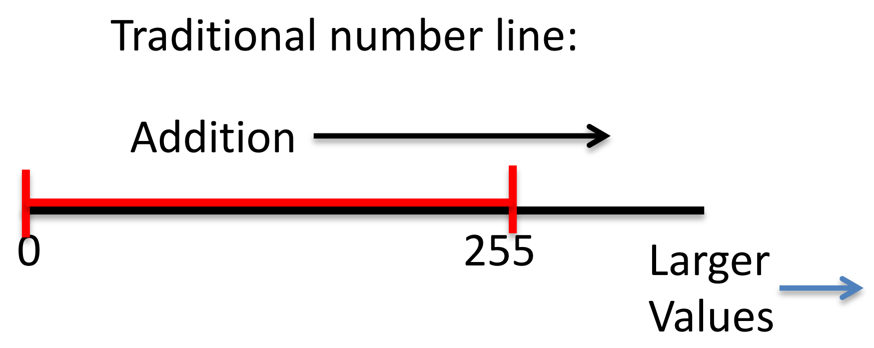
- Suppose we had one byte
  - Can represent $2^8$ (256) values
  - If unsigned (strictly non-negative): 0 – 255

252 = 11111100

253 = 11111101

254 = 11111110

255 = 11111111



# Unsigned Integers

Suppose we had one byte

- Can represent $2^8$ (256) values
- If unsigned (strictly non-negative): 0 – 255

252 = 11111100

253 = 11111101

254 = 11111110

255 = 11111111

What if we add one more?

Car odometer "rolls over".

Any time we are dealing with a finite storage space we cannot represent an infinite number of values!

# Unsigned Integers

Suppose we had one byte

- Can represent $2^8$ (256) values
- If unsigned (strictly non-negative):

  0 – 255

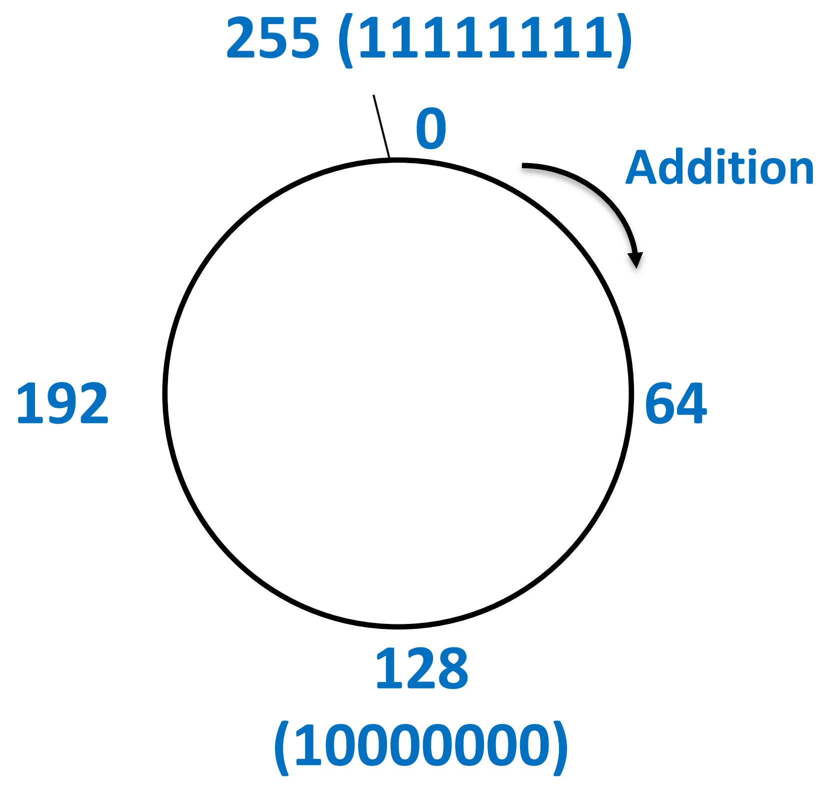
  252 = 11111100

  253 = 11111101

  254 = 11111110

  255 = 11111111

<u>What if we add one more?</u>




Modular arithmetic: Here, all values are modulo 256.

# Unsigned Addition (4-bit)

- Addition works like grade school addition:

```
     1
     0110      6
  +  0100   +  4
  -------   ----
     1010     10
```

Four bits give us range: 0 - 15

# Unsigned Addition (4-bit)

- Addition works like grade school addition:

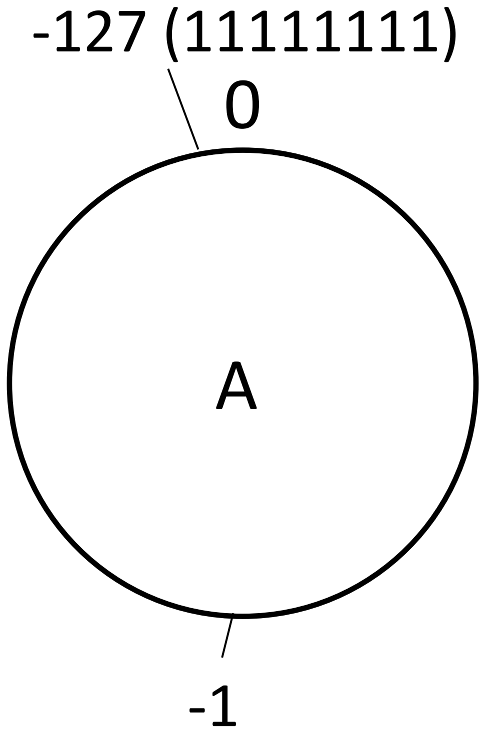
```
   1
   0110      6          1100     12
+  0100   +  4       +  1010    +10
   ----   ----          ----    ---
   1010     10        1 0110      6
^no carry out        ^carry out
```

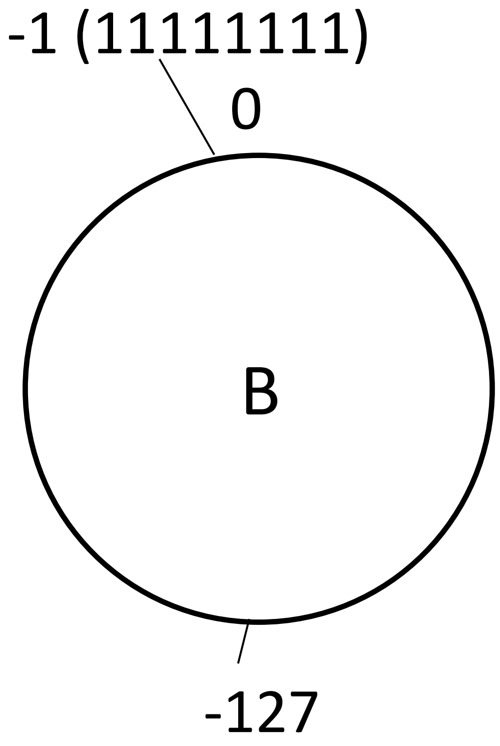
Four bits give us range: 0 - 15

Overflow!

Carry out is indicative of something having gone wrong when adding unsigned values

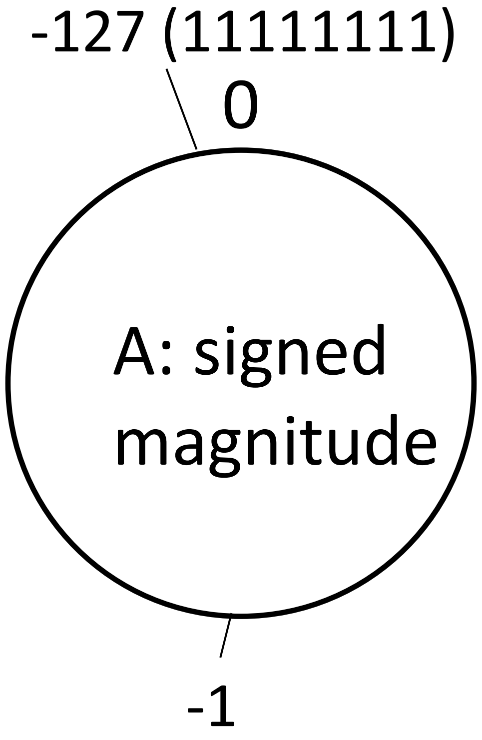
Suppose we want to support signed values (positive <u>and</u> negative) in 8 bits, where should we put -1 and -127 on the circle? Why?

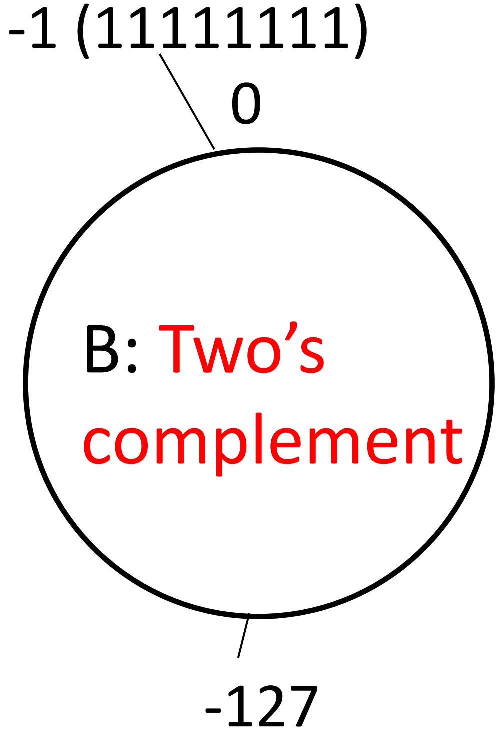



C: Put them somewhere else.

Suppose we want to support signed values (positive and negative) in 8 bits, where should we put -1 and -127 on the circle? Why?




C: Put them somewhere else.

# NOT USED: Signed Magnitude Representation  (for 4 bit values)

- One bit (usually left-most) signals:
  - 0 for positive
  - 1 for negative

This is not what we do in present day systems

For one byte:

1 = 00000001, -1 = 10000001

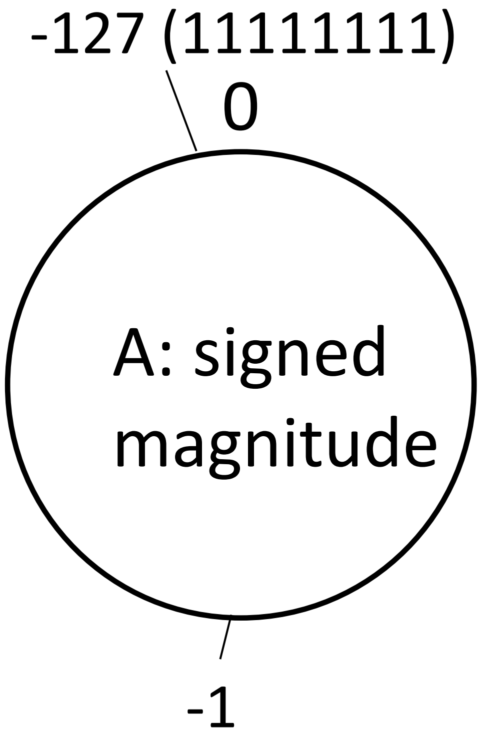
Pros: Negation (negative value of a number) is very simple!

For one byte:

0 = 00000000

What about 10000000?

Major con: Two ways to represent zero!



# Two’s Complement Representation (for four bit values)

Used Today

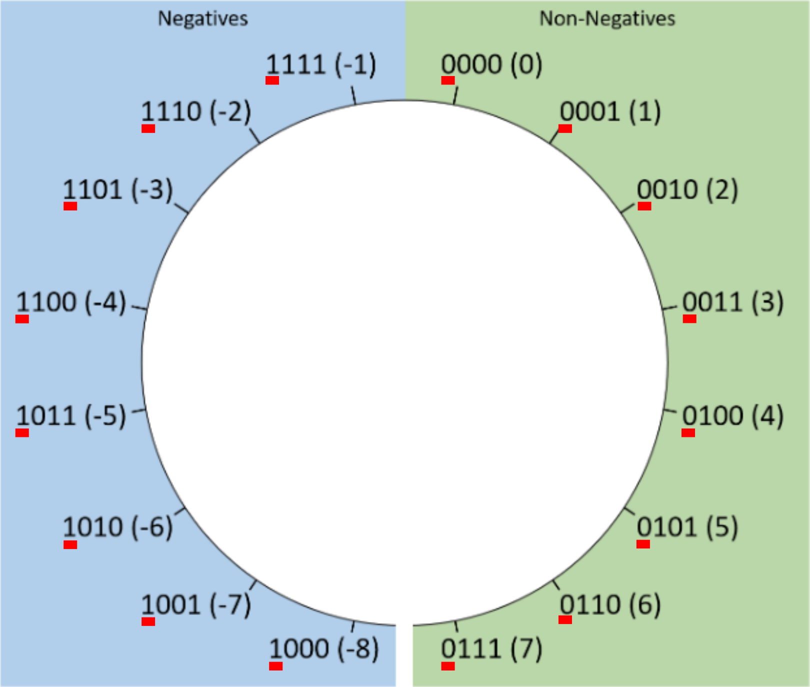
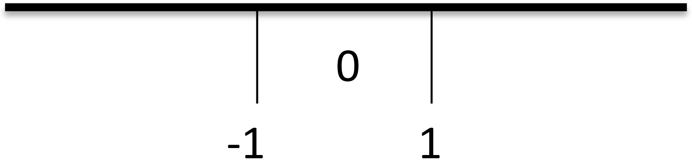



*Figure 2. A logical layout of two’s complement values for bit sequences of length four.*

For an 8 bit range we can express 256 unique values:

- 128 non-negative values (0 to 127)
- 128 negative values (-1 to -128)

- Borrow nice property from number line:




Only one instance of zero!
Implies: -1 and 1 on either side of it.

# Two's Complement

- Only one value for zero
- With N bits, can represent the range:
  - $-2^{N-1}$ to $2^{N-1} - 1$
- Most significant (first) bit still designates positive (0) /negative (1)
- Negating a value is slightly more complicated:

  1 = 00000001,        -1 = 11111111

From now on, unless we explicitly say otherwise, we'll assume all integers are stored using two's complement! This is the standard!

# Two's Compliment

Each two's compliment number is now:

$$[-2^{n-1}*d_{n-1}] \quad + \quad [2^{n-2}*d_{n-2}] \quad +\ldots+ \quad [2^{1}*d_{1}] \quad + \quad [2^{0}*d_{0}]$$

Note the negative sign on just the first digit.
This is why first digit tells us negative vs. positive.

(The other digits are unchanged and carry the same meaning as unsigned.)

# If we interpret 11001 as a two's complement number, what is the value in decimal?

Each two's compliment number is now:

$$[-2^{n-1} * d_{n-1}] \quad + \quad [2^{n-2} * d_{n-2}] \quad + \ldots + \quad [2^{1} * d_{1}] \quad + \quad [2^{0} * d_{0}]$$

A. -2

B. -7

C. -9

D. -25

# If we interpret 11001 as a two's complement number, what is the value in decimal?

Each two's compliment number is now:

$$[-2^{n-1} * d_{n-1}] \quad + \quad [2^{n-2} * d_{n-2}] \quad +\ldots+ \quad [2^{1} * d_{1}] \quad + \quad [2^{0} * d_{0}]$$

A. -2

B. -7      -16 + 8 + 1 = -7

C. -9

D. -25

# “If we interpret…”

- What is the decimal value of 1100?

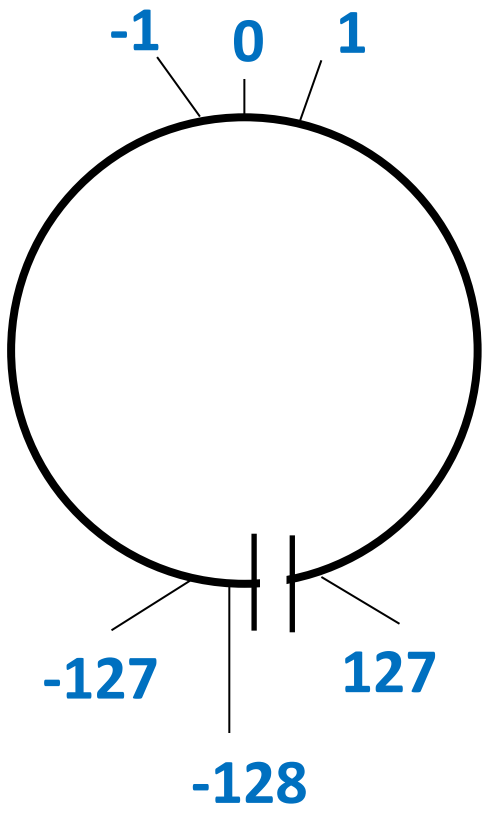
- …as unsigned, 4-bit value: 12  (%u)
- …as signed (two’s complement), 4-bit value: -4  (%d)

- …as an 8-bit value: 12
  (i.e., **0000**1100)

# Two's Complement Negation

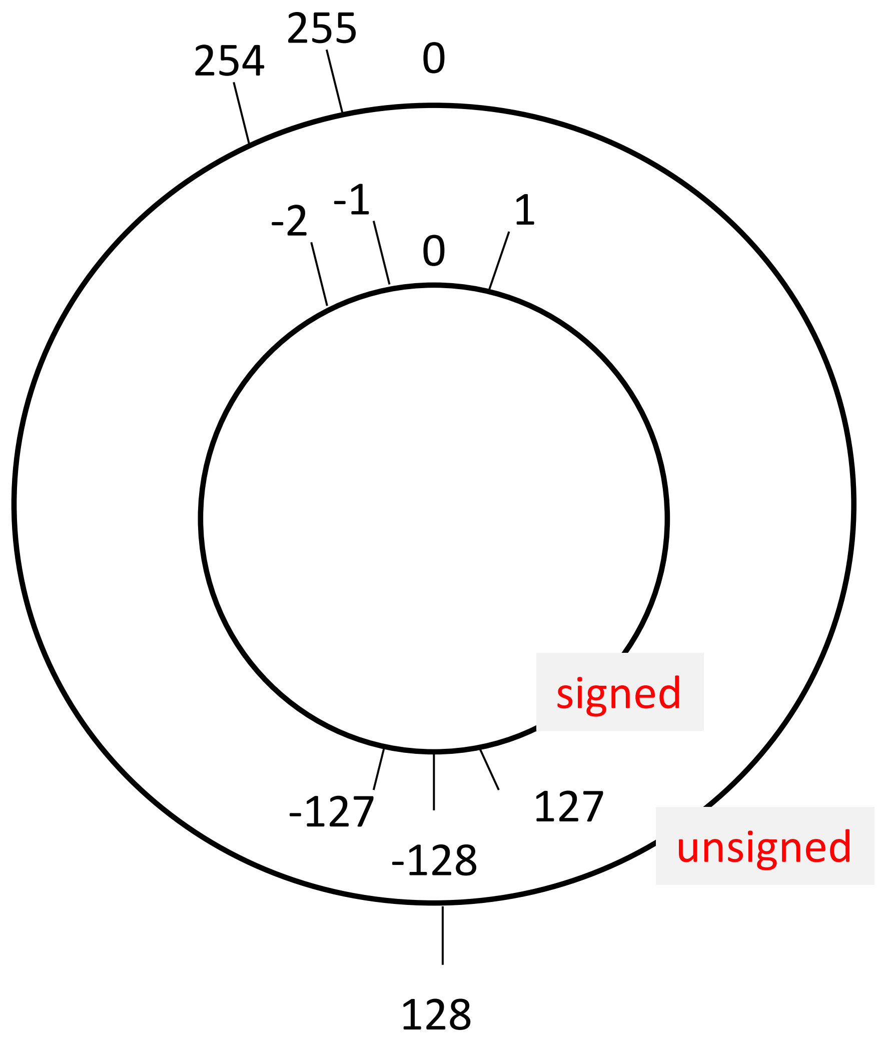
- To negate a value x, we want to find y such that x + y = 0.

- For N bits, $y = 2^N - x$



# Negation Example (8 bits)

- For N bits, $y = 2^N - x$
- Negate 00000010 (2)
  - $2^8 - 2 = 256 - 2 = 254$
- Our wheel only goes to 127!
  - Put -2 where 254 would be if wheel was unsigned.
  - 254 in binary is 11111110




Given 11111110, it's 254 if interpreted as <u>unsigned</u> and -2 interpreted as <u>signed</u>.

# Negation Shortcut

- A much easier, faster way to negate:
  - Flip the bits (0's become 1's, 1's become 0's)
  - Add 1

- Negate 00101110 (46)
  - $2^8$ - 46 = 256 - 46 = 210
  - 210 in binary is 11010010

| | |
|---|---|
| 46: | 00101110 |
| Flip the bits: | 11010001 |
| Add 1 | |
| + 1 | |
| -46: | 11010010 |

# Decimal to Two's Complement with 8-bit values

## (high-order bit is the sign bit)

For positive values, use same algorithm as unsigned

```
For example, 6:   6 - 4 = 2 (4:2^2)
                  2 - 2 = 0 (2:2^1):  00000110
```

For negative values:

1. convert the equivalent positive value to binary
2. then negate binary to get the negative representation

```
For example, -3:
           3: 00000011
       negate: 11111100+1 = 11111101 = -3
```

# What is the 8-bit, two's complement representation for -7?

For negative values:

1. convert the equivalent positive value to binary
2. then negate binary to get the negative representation

A. 11111001
B. 00000111
C. 11111000
D. 11110011

# What is the 8-bit, two’s complement representation for -7?

For negative values:

1. convert the equivalent positive value to binary
2. then negate binary to get the negative representation

A. 11111001
B. 00000111
C. 11111000
D. 11110011

-7 = (1) 7: 00000111
(2) negate: 11111000 + 1 = 11111001

# Addition & Subtraction

- Addition is the same as for unsigned
  - One exception: different rules for overflow
  - Can use the same hardware for both

- Subtraction is the same operation as addition
  - Just need to negate the second operand…

- 6 - 7 = 6 + (-7) = 6 + (~7 + 1)
  - ~7 is shorthand for "flip the bits of 7"

# Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

input 1 ---------------------------------> ADD CIRCUIT

input 2 --> possible bit flipper --> ADD CIRCUIT ---> result

possible +1 input--------> ADD CIRCUIT

Let's call this possible +1 input: "Carry in"

(0: on add, 1: on subtract)

# 4-bit signed Examples:

Subtraction via Addition:

– a-b is same as a + ~b + 1

Subtraction: flip bits and add 1

```
3 -  6 =    0011
             1001      (6: 0110  ~6: 1001)
           +    1
             1101 = -3
```

Addition:

```
3 + -6 =    0011
           + 1010
             1101 = -3
```

# Signed & Unsigned 4-bit Subtraction:

Unsigned subtraction: flip bits and add 1

```
13 -  1 =
```

Signed subtraction: flip bits and add 1

```
-3 - 1 =
```

A. 1100 & 1100
B. 1100 & 1010
C. 1010 & 1010
D. 1001 & 1100

# Signed & Unsigned 4-bit Subtraction:

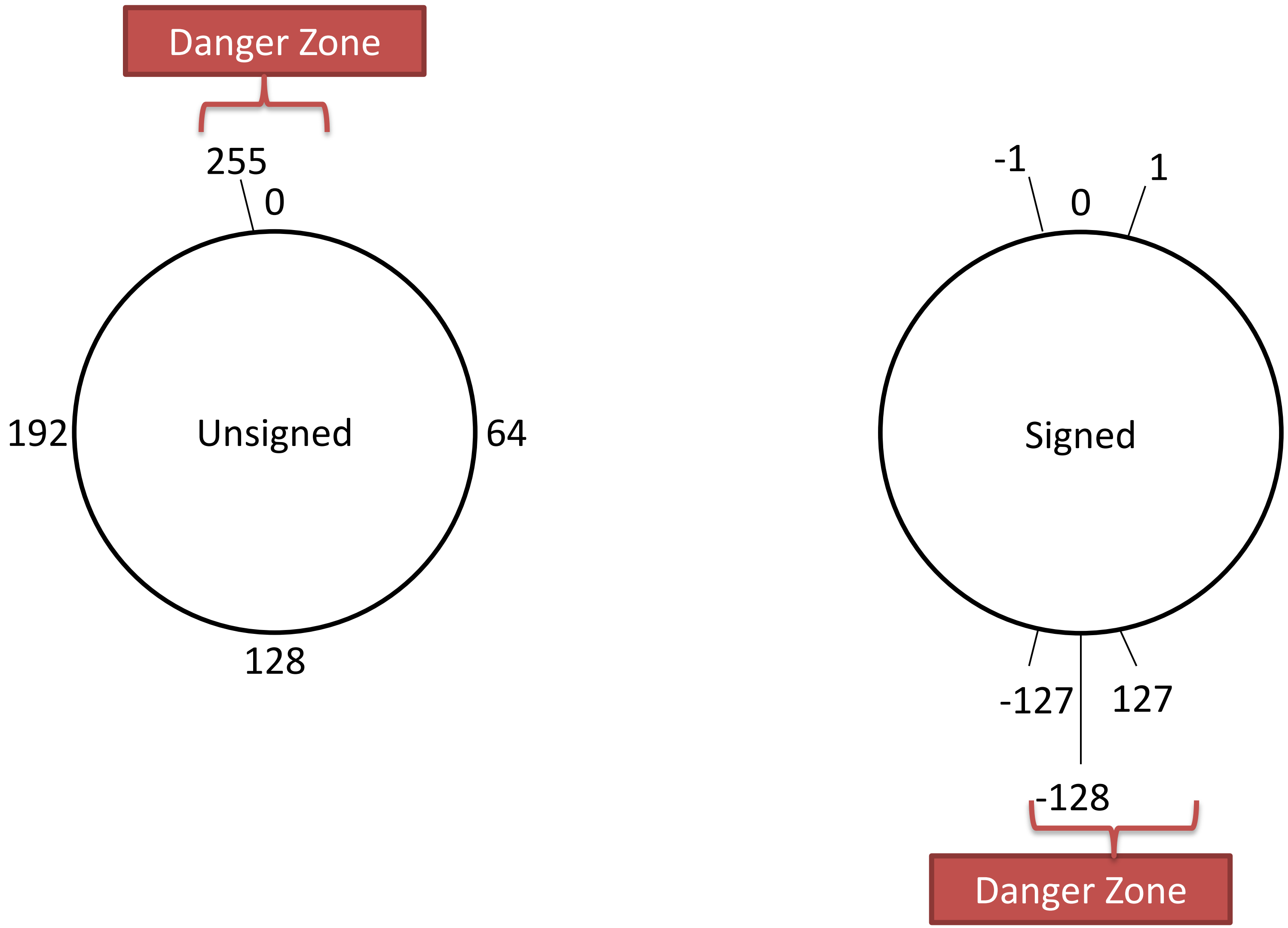
## Unsigned subtraction: flip bits and add 1

```
13 -   1 =   1101
             1110      (1: 0001   ~1: 1110)
           + ____1
         1   1100 = 12
```

## Signed subtraction: flip bits and add 1

```
-3 - 1 =     1101
             1110
           + ____1
         1   1100 = -4
```

# Overflow, Revisited

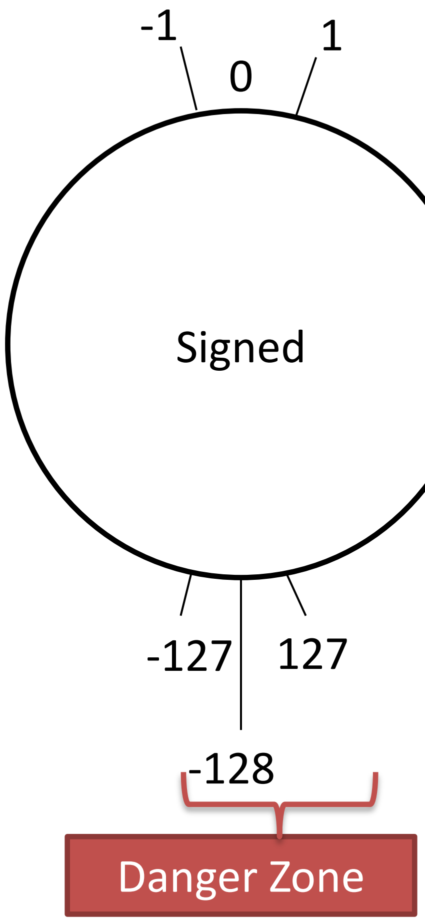


# If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always

B. Sometimes

C. Never

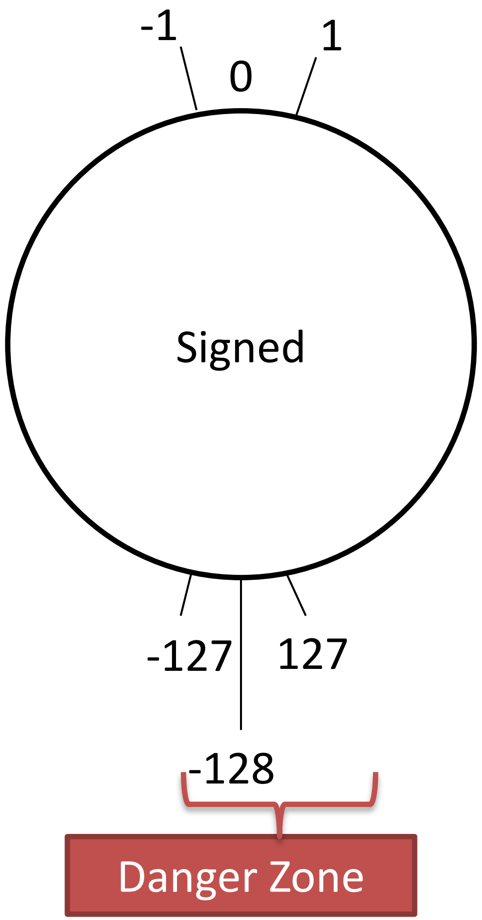


# If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always

B. Sometimes

C. Never



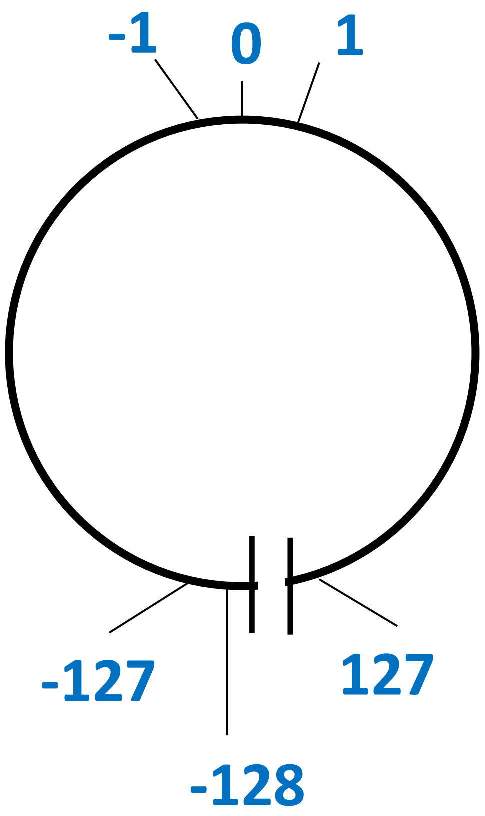
# Two's Complement Overflow For Addition

- **Addition Overflow**: IFF the sign bits of operands are the same, but the sign bit of result is different.
- Not enough bits to store result!

**sign of operands = sign of result**

no overflow

```
 3+4=7      -2+-3=-5
 0011        1110
+0100       +1101
 0111      1 1011
```



# Two's Complement Overflow For Addition

- **Addition Overflow**: IFF the sign bits of operands are the same, but the sign bit of result is different.
- Not enough bits to store result!

**sign of operands = sign of result**

no overflow

```
 3+4=7      -2+-3=-5
 0011          1110
+0100         +1101
 0111        1 1011
```

**sign of operands ≠ sign of result**

overflow

```
 4+7=11     -6-8=-14
 0100          1010
+0111         +1000
 1011        1 0010
```

# Recall: Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

```
input 1 --------------------------------> +-------------+
input 2 --> | possible bit flipper | -->  | ADD CIRCUIT | ---> result
            possible +1 input-------->    +-------------+
```

Let’s call this possible +1 input: “Carry in”

(0: on add, 1: on subtract)

# How many of these unsigned operations have overflowed?

Interpret these as 4-bit unsigned values (valid range 0 to 15):

Addition (carry-in = 0)

| | | | carry-in | carry-out | |
|---|---|---|---|---|---|
| 9 + 11 = | 1001 + | 1011 + | 0 = | 1 | 0100 |
| 9 + 6 = | 1001 + | 0110 + | 0 = | 0 | 1111 |
| 3 + 6 = | 0011 + | 0110 + | 0 = | 0 | 1001 |

Subtraction (carry-in = 1)

| | | | | | |
|---|---|---|---|---|---|
| 6 - 3 = | 0110 + | $\overbrace{1100}^{(-3)}$ + | 1 = | 1 | 0011 |
| 3 - 6 = | 0011 + | $\underbrace{1001}_{(-6)}$ + | 1 = | 0 | 1101 |

A. 1
B. 2
C. 3
D. 4
E. 5

# How many of these unsigned operations have overflowed?

Interpret these as 4-bit unsigned values (valid range 0 to 15):

Notice a Pattern?

Addition (carry-in = 0)

| | | | carry-in | carry-out | | |
|---|---|---|---|---|---|---|
| 9 + 11 = | 1001 + | 1011 + | 0 = | 1 | 0100 = | 4 |
| 9 + 6 = | 1001 + | 0110 + | 0 = | 0 | 1111 = | 15 |
| 3 + 6 = | 0011 + | 0110 + | 0 = | 0 | 1001 = | 9 |

Subtraction (carry-in = 1)

$$6 - 3 = 0110 + \overbrace{1100}^{(-3)} + 1 = 1 \quad 0011 = 3$$

$$3 - 6 = 0011 + \underbrace{1001}_{(-6)} + 1 = 0 \quad 1101 = 13$$

A. 1
B. 2
C. 3
D. 4
E. 5

# How many of these unsigned operations have overflowed?

Interpret these as 4-bit unsigned values (valid range 0 to 15):

Notice a Pattern?

Addition (carry-in = 0)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | carry-in | | carry-out | | | |
| 9 + 11 = | 1001 | + | 1011 | + | 0 | = | 1 | 0100 | = | 4 |
| 9 + 6 = | 1001 | + | 0110 | + | 0 | = | 0 | 1111 | = | 15 |
| 3 + 6 = | 0011 | + | 0110 | + | 0 | = | 0 | 1001 | = | 9 |

Subtraction (carry-in = 1)

$$6 - 3 = 0110 + \overbrace{1100}^{(-3)} + 1 = 1\ \ 0011 = 3$$

$$3 - 6 = 0011 + \underbrace{1001}_{(-6)} + 1 = 0\ \ 1101 = 13$$

A. 1
B. 2
C. 3
D. 4
E. 5

# Overflow Rule Summary

Unsigned: overflow

- The carry-in bit is different from the carry-out.

| $C_{in}$ | $C_{out}$ | $C_{in}$ XOR $C_{out}$ |
|---|---|---|
| 0 | 0 | 0 |
| **0** | **1** | **1** |
| **1** | **0** | **1** |
| 1 | 1 | 0 |

# Two's Complement Overflow For Subtraction

**Subtraction Overflow Rules Summarized**:

- Overflow occurs IFF the sign bits of the subtraction operands are different, and the sign bit of the Result and Subtrahend are the same as shown below:
  - Minuend - Subtrahend = Result
  - If positive – negative = negative (overflow)
  - If negative – positive = positive (overflow)

# Two's Complement Overflow For Subtraction

– **Rule 1:**

Minuend | Subtrahend | Result

- Positive operand - Negative operand = Positive Result: No Overflow
- Positive operand - Negative operand = Negative Result: Overflow
- **Intuition:** We know a positive – negative is equivalent to a positive + positive.
  - *If this sum does not result in a positive value we have an overflow*

Subtrahend and Result have **different sign bits**

**no overflow**

```
2-(-3)=5      3-(-4)=7
 0010          0011
-1110         -1100

 0010          0011
+0011         +0100
 0101          0111
```

Subtrahend and Result have the **same sign bits**

**overflow**

```
2-(-6)=8           3-(-7)=10
 0010               0011
-1010              -1001

 0010               0011
+0110              +0111
 1000 (-8)          1010 (-6)
```

# Two's Complement Overflow For Subtraction

– **Rule 2:**

Minuend  Subtrahend  Result

- Negative operand - Positive operand = Negative Result: No Overflow
- Negative operand - Positive operand = Positive Result:  Overflow
- **Intuition:** We know a negative – positive number is equivalent to a negative + negative number.
  – *If this sum does not result in a negative value we have an overflow*

Subtrahend and Result have **different sign bits**

no overflow

```
-2-(3)=-5        -3-(4)=-7
  1110             1101
 -0011            -0100

  1110             1101
 +1101            +1100
1 1011(-5)       1 1001(-7)
```

Subtrahend and Result have the **same sign bits**

overflow

```
-2-(7)=-9        -4-(7)=-11
  1110             1100
 -0111            -0111

  1110             1100
 +1001            +0111
1 0111(7)        1 0011(-6)
```

# Two's Complement Overflow For <u>Subtraction</u>

- **Rule 1:**

  Minuend    Subtrahend    Result

  - Positive operand - Negative operand = Positive Result: No Overflow
  - Positive operand - Negative operand = Negative Result: Overflow
  - **Intuition:** We know a positive – negative is equivalent to a positive + positive.
    - *If this sum does not result in a positive value we have an overflow*

- **Rule 2:**

  Minuend    Subtrahend    Result

  - Negative operand - Positive operand = Negative Result: No Overflow
  - Negative operand - Positive operand = Positive Result: Overflow
  - **Intuition:** We know a negative – positive number is equivalent to a negative + negative number.
    - *If this sum does not result in a negative value we have an overflow*

# Overflow Rule Summary

- Signed overflow:
  - The sign bits of operands are the same, but the sign bit of result is different.

- Unsigned: overflow
  - The carry-in bit is different from the carry-out.

| $C_{in}$ | $C_{out}$ | $C_{in}$ XOR $C_{out}$ |
|---|---|---|
| 0 | 0 | 0 |
| **0** | **1** | **1** |
| **1** | **0** | **1** |
| 1 | 1 | 0 |

So far, all arithmetic on values that were the same size. What if they're different?

# Sign Extension

When combining signed values of different sizes, expand the smaller value to equivalent larger size:

```
char y = 2, x = -13;
short z = 10;
```

```
   z = z + y;                          z = z + x;

0000000000001010                    0000000000000101
+       00000010                    +       11110011
0000000000000010                    1111111111110011
```

Fill in high-order bits with sign-bit value to get same numeric value in larger number of bytes.

# Let's verify that this works

4-bit signed value, sign extend to 8-bits, is it the same value?

| | | | |
|---|---|---|---|
| `0111` | `--->` | `0000 0111` | obviously still 7 |
| `1010` | `--->` | `1111 1010` | is this still -6? |

-128 + 64 + 32 + 16 + 8 + 0 + 2 + 0 = -6 yes!

# Operations on Bits

- For these, it doesn't matter how the bits are interpreted (signed vs. unsigned)

- Bit-wise operators (AND, OR, NOT, XOR)

- Bit shifting

# Bit-wise Operators

- Bit operands, Bit result (interpret as appropriate for the context)

& (AND)     | (OR)     ~(NOT)     ^(XOR)

| A | B | A & B | A \| B | ~A | A ^ B |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 |

```
  01101010      01010101      10101010     ~10101111
& 10111011    | 00100001    ^ 01101001      --------
  --------      --------      --------      01010000
  00101010      01110101      11000011
```

# More Operations on Bits (Shifting)

Bit-shift operators: << left shift, >> right shift

```
01010101 << 2  is 01010100
                  2 high-order bits shifted out
                  2 low-order bits filled with 0
01101010 << 4  is 10100000
01010101 >> 2  is 00010101
01101010 >> 4  is 00000110

10101100 >> 2  is 00101011 (logical shift)
               or 11101011 (arithmetic shift)
```

Arithmetic right shift: fills high-order bits w/sign bit

C automatically decides which to use based on type: signed: arithmetic, unsigned: logical

# Try some 4-bit examples:

bit-wise operations:

- 0101 & 1101
- 0101 | 1101

Logical (unsigned) bit shift:

- 1010 << 2
- 1010 >> 2

Arithmetic (signed) bit shift:

- 1010 << 2
- 1010 >> 2

# Try some 4-bit examples:

bit-wise operations:

- 0101 & 1101 = 0101
- 0101 | 1101 = 1101

Logical (unsigned) bit shift:

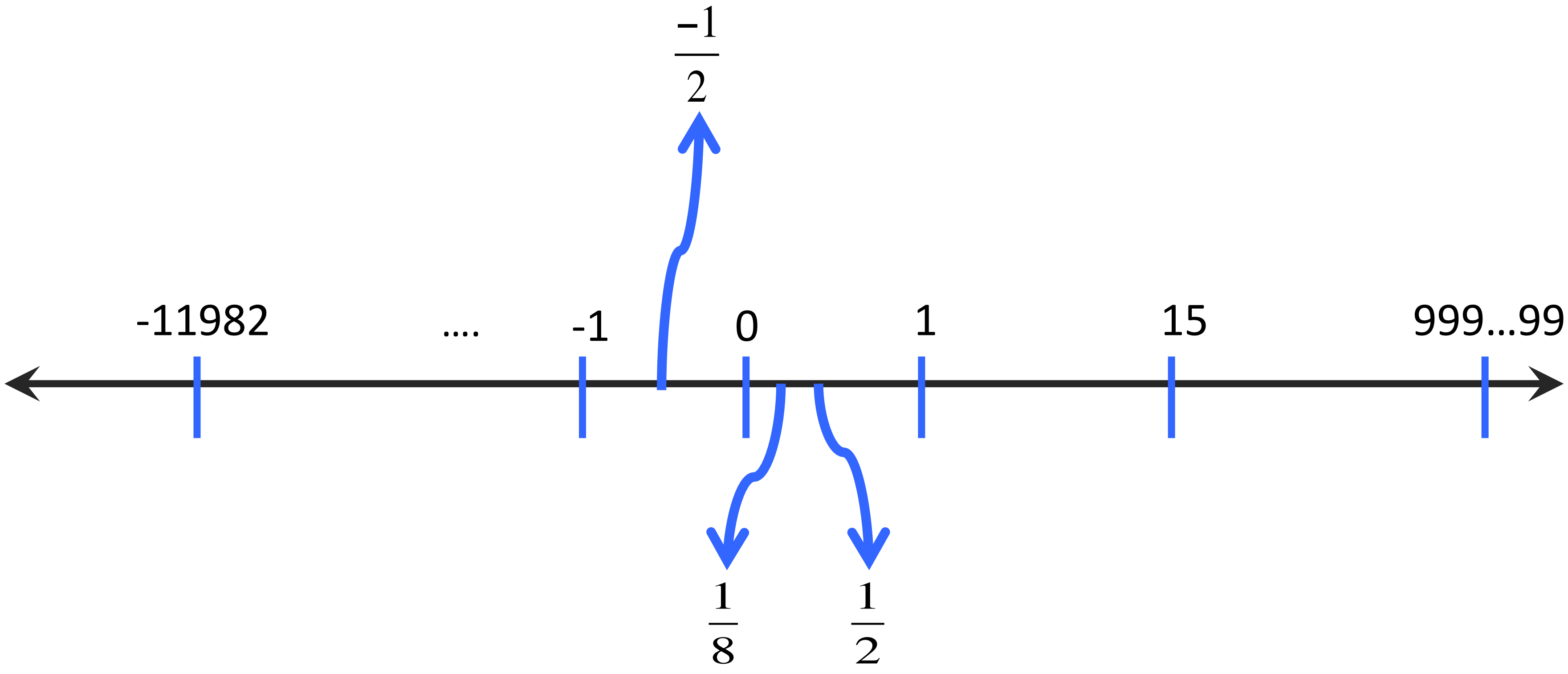
- 1010 << 2 = 1000
- 1010 >> 2 = 0010

Arithmetic (signed) bit shift:

- 1010 << 2 = 1000
- 1010 >> 2 = 1110

# Additional Info: (not assessable) Fractional binary numbers

How do we represent fractions in binary?



# Additional Info: (not assessable) Floating Point Representation

1 bit for sign　　　sign | exponent | fraction |

8 bits for exponent

23 bits for precision

$$value = (-1)^{sign} * 1.fraction * 2^{(exponent-127)}$$

let's just plug in some values and try it out

```
0x40ac49ba: 0 10000001   01011000100100110111010
     sign = 0 exp = 129   fraction = 2902458

        = 1*1.2902458*2^2 = 5.16098
```

<u>You're not expected to memorize this</u>

# Summary

- Images, Word Documents, Code, and Video can represented in bits.
- Byte or 8 bits is the smallest addressable unit
- N bits can represent $2^N$ <u>unique</u> values
- A number is written as a sequence of digits: in the decimal base system
  - [dn * 10 ^ n] + [dn-1 * 10 ^ n-1] + ... + [d2 * 10 ^ 2] + [d1 * 10 ^ 1] + [d0 * 10 ^ 0]
  - For any base system:
  - [dn * b ^ n] + [dn-1 * b ^ n-1] + ... + [d2 * b ^ 2] + [d1 * b ^ 1] + [d0 * b ^ 0]
- Hexadecimal values (represent 16 values): {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}
  - Each hexadecimal value can be represented by 4 bits. (2^4=16)
- <u>A finite storage space we cannot represent an infinite number of values</u>. For e.g., the max unsigned 8 bit value is 255.
  - Trying to represent a value >255 will result in an overflow.
- Two's Complement Representation: 128 non-negative values (0 to 127), and 128 negative values (-1 to -128).